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Inverse Hyperbolic Conduction Problem in Estimating Two Unknown Surface Heat Fluxes Simultaneously

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An inverse hyperbolic heat conduction problem is solved in the present study by an iterative regularization method, that is, the conjugate gradient method, to estimate simultaneously two unknown boundary heat fluxes based on the interior temperature measurements. The inverse solutions will be justified based on the numerical experiments in which two specific cases in determining the unknown boundary heat flux distributions are examined. Results show that the inverse solutions can always be obtained with any arbitrary initial guesses of the boundary heat fluxes and that the position of each sensor should be as close to each boundary as possible to obtain accurate estimations. Finally, it is concluded that accurate boundary heat fluxes can be estimated in the present study when large measurement errors are considered.

Nomenclature

c	=	heat capacity
J	=	functional defined by Eq. (2)
J'_1, J'_2	=	gradient of functional defined by Eqs. (15) and (16)
k	=	thermal conductivity
P_1, P_2	=	directions of descent defined by Eq. (4)
q_1, q_2	=	unknown surface heat fluxes
T	=	calculated temperature
Y	=	measured temperatures
β_1, β_2	=	search step sizes defined by Eq. (10)
γ_1, γ_2	=	conjugate coefficients defined by Eq. (5)
ΔT, $\Delta\tilde{T}$	=	solution for the sensitivity problems defined by Eqs. (6) and (7)
ε	=	convergence criteria
λ_1, λ_2	=	solutions for the adjoint problem
ρ	=	density
σ	=	standard deviation of the measurement errors
τ	=	relaxation time
ω	=	random number

Superscript

$\hat{}$	=	estimated values

I. Introduction

HYPERBOLIC heat transport has been receiving increasing attention both for theoretical motivations and for the analysis of some practical problems involving a fast supply of thermal energy, for instance, by a laser pulse or a chemical explosion, etc.

Cattaneo [1] and Vernotte [2] suggested the hyperbolic heat equation model for the heat transfer with a finite propagation speed. The hyperbolic heat equation is often employed to study the temperature fields and related heat transfer quantities in transient heat flow in an extremely short period of time at a very high temperature gradient or for very low temperatures near absolute zero. When solving this equation, one may encounter some mathematical difficulties that must be precluded to obtain an admissible solution. Various numerical methods [3–8] have been proposed to solve the hyperbolic heat conduction problems under different applications and in different configurations.

In Taitel's analysis [8], the temperature overshoot problem was experienced under some special conditions, and the method of discrete formulation (or difference equation) is suggested to avoid such a temperature overshoot problem. In the present work, the technique of a central difference equation, as suggested by Carey and Tsai [4], is applied and therefore the temperature overshoot problem can be avoided.

The direct hyperbolic heat conduction problems are concerned with the determination of temperature at interior points of a region when the initial and boundary conditions, thermophysical properties, and heat generation are specified. In contrast, the inverse hyperbolic heat conduction problem considered here involves the determination of two surface heat fluxes simultaneously from the knowledge of the interior temperature measurements taken inside the material.

The study on the inverse hyperbolic heat conduction problem is limited in the literature. The hybrid method of the Laplace transform technique and the finite-difference method were used by Chen and Chang [9] to estimate the unknown surface temperature based on measured temperatures. The similar algorithm with a sequential-in-time concept was applied by Chen et al. [10] to estimate the unknown surface temperature in a two-dimensional inverse problem. Chen et al. [11] used the Laplace transform technique and control method in conjunction with the hyperbolic shape function and least-square method in a one-dimensional inverse hyperbolic heat conduction problem in estimating the boundary condition.

Yang [12] applied the modified Newton–Raphson method with the concept of future time to determine the boundary heat flux in a one-dimensional hyperbolic heat conduction problem. The drawbacks of the estimated inverse solutions are 1) there always exists a phase error between the exact and estimated heat flux, even when the exact measurement (i.e., no measurement error) is considered; and 2) the inverse solutions are very sensitive to the measurements because the chosen standard deviation for the measurement is always very small in comparison with the measured temperatures.

The technique of conjugate gradient method (CGM) [13] has shown its potential for solving many kinds of inverse problems and has been applied to many different applications. For instance, Huang and Huang [14] used the CGM in an inverse biotechnology problem to estimate the optical diffusion and absorption coefficients of tissue simultaneously. Huang and Shih [15] applied the CGM to estimate the interfacial configurations for a multiple region domain. Huang and Lo [16] examined a three-dimensional inverse problem in predicting the distribution of heat fluxes in the cutting tools by using the steepest descent method (SDM). Huang and Wu [17] applied the iterative regularization method in estimating the base temperature for

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non-Fourier fins. Huang and Chen [18] studied the inverse phonon radiative transport problem in estimating the boundary temperatures for a nanoscale thin film.

Recently, Huang and Wu [19] applied the CGM to a similar problem that was examined by Yang [12] and found that the drawbacks for [12] can be avoided and the accuracy of the inverse solutions can be improved. The objective of the present inverse study is to extend the previous work by Huang and Wu [19], but now both boundary heat fluxes are assumed unknown. The goal is to use the CGM to estimate simultaneously two unknown boundary heat fluxes based on the interior temperature measurements.

The CGM is also called an iterative regularization method, which means that the regularization procedure is performed during the iterative processes and, thus, the determination of optimal regularization conditions is not needed. It is derived from the perturbation principles and has transformed the inverse problem to the solution of three problems, namely, the direct, the sensitivity, and the adjoint problem, which are discussed in detail in the following sections.

II. Direct Problem

The following slab problem with thickness L and constant thermal properties is considered to illustrate the methodology for developing expressions for use in determining simultaneously two unknown boundary heat fluxes in the present inverse hyperbolic problem. The initial temperature equals to T_0. When $t > 0$, the boundary conditions at boundaries $x = 0$ and L are subjected to the unknown heat fluxes $q_1(t)$ and $q_2(t)$, respectively.

The mathematical formulation of this linear hyperbolic heat conduction problem is given by

$$k\frac{\partial^2 T}{\partial x^2} = \tau\rho c\frac{\partial^2 T}{\partial t^2} + \rho c\frac{\partial T}{\partial t}; \qquad 0 < x < L, \qquad t > 0 \tag{1a}$$

$$T(x,0) = T_0; \qquad 0 < x < L, \qquad t = 0 \tag{1b}$$

$$\frac{\partial T(x,0)}{\partial t} = 0; \qquad 0 < x < L, \qquad t = 0 \tag{1c}$$

$$k\frac{\partial T(0,t)}{\partial x} = q_1(t) + \tau\frac{\partial q_1(t)}{\partial t}; \qquad x = 0, \qquad t > 0 \tag{1d}$$

$$k\frac{\partial T(L,t)}{\partial x} = q_2(t) + \tau\frac{\partial q_2(t)}{\partial t}; \qquad x = L, \qquad t > 0 \tag{1e}$$

Here, k, ρ, c, and τ are the thermal conductivity, density, heat capacity, and relaxation time, respectively. The solution for the preceding hyperbolic heat conduction problem can be solved by using the central difference (Crank–Nicolson type) [4]. The direct problem considered here is concerned with the determination of the medium temperature when all the boundary conditions at all boundaries are known.

III. Inverse Problem

For the inverse problem considered here, the boundary heat fluxes at $x = 0$ and L are regarded as being unknown, but everything else in Eqs. (1) is known. In addition, temperature readings at $x = x_1$ and x_2 are considered available.

Letting the temperature readings taken by sensors at $x = x_1$ and x_2 be denoted by $Y(x_1, t)$ and $Y(x_2, t)$, it is noted that the measured temperatures $Y(x_1, t)$ and $Y(x_2, t)$ should contain measurement errors. The inverse problem can thus be stated as follows: by using the previously mentioned measured temperature data $Y(x_1, t)$ and $Y(x_2, t)$, estimate simultaneously two unknown boundary heat fluxes $q_1(t)$ and $q_2(t)$.

The solution of this inverse problem is to be obtained in such a way that the following functional is minimized:

$$J[q_1(t), q_2(t)] = \int_{t=0}^{t_f} \{[T(x_1,t) - Y(x_1,t)]^2 + [T(x_2,t) - Y(x_2,t)]^2\}\,\mathrm{d}t \tag{2}$$

Here, $T(x_1, t)$ and $T(x_2, t)$ are the estimated or computed temperatures at $x = x_1$ and x_2, respectively, with time t, and t_f is the final measurement time. They can be obtained from the solution of the direct problem given previously by using estimated heat fluxes for the exact $q_1(t)$ and $q_2(t)$.

IV. Conjugate Gradient Method for Minimization

The following iterative process based on the conjugate gradient method [13] is now used for the estimation of unknown heat fluxes $q_1(t)$ and $q_2(t)$ by minimizing the functional $J[q_1(t), q_2(t)]$:

$$q_1^{n+1}(t) = q_1^n(t) - \beta_1^n P_1^n(t); \quad \text{for } n = 0, 1, 2, \ldots \tag{3a}$$

$$q_2^{n+1}(t) = q_2^n(t) - \beta_2^n P_2^n(t); \quad \text{for } n = 0, 1, 2, \ldots \tag{3b}$$

where β_1^n and β_2^n are the search step sizes in going from iteration n to iteration $n + 1$, and $P_1^n(t)$ and $P_2^n(t)$ are the directions of descent (i.e., search directions) given by

$$P_1^n(t) = J_1'^n(t) + \gamma_1^n P_1^{n-1}(t) \tag{4a}$$

$$P_2^n(t) = J_2'^n(t) + \gamma_2^n P_2^{n-1}(t) \tag{4b}$$

which are the conjugation of the gradient directions $J_1'^n(t)$ and $J_2'^n(t)$ at iteration n and the directions of descent $P_1^{n-1}(t)$ and $P_2^{n-1}(t)$ at iteration $n - 1$. The conjugate coefficients can be determined from

$$\gamma_1^n = \frac{\int_{t=0}^{t_f} (J_1'^n)^2\,\mathrm{d}t}{\int_{t=0}^{t_f} (J_1'^{n-1})^2\,\mathrm{d}t} \quad \text{with} \quad \gamma_1^0 = 0 \tag{5a}$$

$$\gamma_2^n = \frac{\int_{t=0}^{t_f} (J_2'^n)^2\,\mathrm{d}t}{\int_{t=0}^{t_f} (J_2'^{n-1})^2\,\mathrm{d}t} \quad \text{with} \quad \gamma_2^0 = 0 \tag{5b}$$

It should be noted that when $\gamma_1^n = \gamma_2^n = 0$ for any n in Eqs. (5), the directions of descent $P_1^n(t)$ and $P_2^n(t)$ become the gradient directions, that is, the SDM is obtained. The convergence of the CGM in minimizing the functional J is guaranteed in [20].

To execute the iterations according to Eqs. (3), two step sizes β_1^n and β_2^n and two gradient equations $J_1'^n(t)$ and $J_2'^n(t)$ must be calculated. To develop expressions for the determination of these quantities, two sensitivity problems and an adjoint problem need to be constructed, as described as follows.

A. Sensitivity Problems and Search Step Sizes

The present inverse problem involves two unknown functions $q_1(t)$ and $q_2(t)$; to derive the sensitivity problem for each unknown function, we should perturb the unknown functions one at a time.

First, it is assumed that when $q_1(t)$ undergoes a variation $\Delta q_1(t)$, $T(x, t)$ are perturbed by $\Delta\bar{T}(x, t)$. Replacing in the direct problem $q_1(t)$ by $q_1(t) + \Delta q_1(t)$, $T(x, t)$ by $T(x, t) + \Delta\bar{T}(x, t)$, subtracting the resulting expressions from the direct problem, and neglecting the second-order terms, the following sensitivity problem for the sensitivity function $\Delta\bar{T}(x, t)$ can be obtained:

$$k\frac{\partial^2 \Delta\bar{T}}{\partial x^2} = \tau\rho c\frac{\partial^2 \Delta\bar{T}}{\partial t^2} + \rho c\frac{\partial \Delta\bar{T}}{\partial t}; \qquad 0 < x < L, \qquad t > 0 \tag{6a}$$

$$\Delta\bar{T}(x,0) = 0; \qquad 0 < x < L, \qquad t = 0 \tag{6b}$$

$$\frac{\partial \Delta \bar{T}(x,0)}{\partial t} = 0; \qquad 0 < x < L, \qquad t = 0 \tag{6c}$$

$$k\frac{\partial \Delta \bar{T}(0,t)}{\partial x} = \Delta q_1(t) + \tau \frac{\partial \Delta q_1(t)}{\partial t}; \qquad x = 0, \qquad t > 0 \tag{6d}$$

$$\frac{\partial \Delta \bar{T}(L,t)}{\partial x} = 0; \qquad x = L, \qquad t > 0 \tag{6e}$$

Similarly, by perturbing $q_2(t)$ with $\Delta q_2(t)$ and $T(x,t)$ with $\Delta\tilde{T}(x,t)$, the second sensitivity problem can be obtained as

$$k\frac{\partial^2 \Delta \tilde{T}}{\partial x^2} = \tau\rho c\frac{\partial^2 \Delta \tilde{T}}{\partial t^2} + \rho c\frac{\partial \Delta \tilde{T}}{\partial t}; \qquad 0 < x < L, \qquad t > 0 \tag{7a}$$

$$\Delta \tilde{T}(x,0) = 0; \qquad 0 < x < L, \qquad t = 0 \tag{7b}$$

$$\frac{\partial \Delta \tilde{T}(x,0)}{\partial t} = 0; \qquad 0 < x < L, \qquad t = 0 \tag{7c}$$

$$\frac{\partial \Delta \tilde{T}(0,t)}{\partial x} = 0; \qquad x = 0, \qquad t > 0 \tag{7d}$$

$$k\frac{\partial \Delta \tilde{T}(L,t)}{\partial x} = \Delta q_2(t) + \tau \frac{\partial \Delta q_2(t)}{\partial t}; \qquad x = L, \qquad t > 0 \tag{7e}$$

The technique of the Crank–Nicolson type central difference discretization [4] is used to solve the two previously mentioned sensitivity problems.

The functional $J(q_1^{n+1}, q_2^{n+1})$ at iteration $n+1$ is obtained by rewriting Eq. (2) as

$$J[q_1(t), q_2(t)] = \int_{t=0}^{t_f} [T(x_1, t; q_1^n - \beta_1^n P_1^n, q_2^n - \beta_2^n P_2^n) - Y(x_1,t)]^2\, \mathrm{d}t + \int_{t=0}^{t_f} [T(x_2, t; q_1^n - \beta_1^n P_1^n, q_2^n - \beta_2^n P_2^n) - Y(x_2,t)]^2\, \mathrm{d}t \tag{8}$$

where $q_1^{n+1}(t)$ and $q_2^{n+1}(t)$ have been replaced by the expressions given by Eqs. (3). If the estimated temperatures $T(x_1, t; q_1^n - \beta_1^n P_1^n, q_2^n - \beta_2^n P_2^n)$ and $T(x_2, t; q_1^n - \beta_1^n P_1^n, q_2^n - \beta_2^n P_2^n)$ are linearized by a Taylor expansion, Eq. (8) takes the form

$$J[q_1(t), q_2(t)] = \int_{t=0}^{t_f} [T(x_1, t; q_1^n, q_2^n) - \beta_1^n \Delta\bar{T}(x_1, t; P_1^n) - \beta_2^n \Delta\tilde{T}(x_1, t; P_2^n) - Y(x_1,t)]^2\, \mathrm{d}t + \int_{t=0}^{t_f} [T(x_2, t; q_1^n, q_2^n) - \beta_1^n \Delta\bar{T}(x_2, t; P_1^n) - \beta_2^n \Delta\tilde{T}(x_2, t; P_2^n) - Y(x_2,t)]^2\, \mathrm{d}t \tag{9}$$

where $T(x_1, t; q_1^n, q_2^n)$ and $T(x_2, t; q_1^n, q_2^n)$ are the solutions of the direct problem by using estimate heat fluxes for exact values at x_1 and x_2 with time t. The sensitivity functions $\Delta\bar{T}(x_1, t; P_1^n)$, $\Delta\bar{T}(x_2, t; P_1^n)$, $\Delta\tilde{T}(x_1, t; P_2^n)$, and $\Delta\tilde{T}(x_2, t; P_2^n)$ are taken as the solutions for problems (6) and (7) at x_1 and x_2 with time t by letting $\Delta q_1(t) = P_1^n(t)$ and $\Delta q_2(t) = P_2^n(t)$, respectively [19].

Equation (9) is differentiated with respect to β_1^n and β_2^n, respectively, and equating them equal to zero to obtain two independent equations. After solving these two equations, two search step sizes β_1^n and β_2^n can be determined as

$$\beta_1^n = (C_3C_5 - C_2C_4)/(C_3C_3 - C_1C_2) \tag{10a}$$

$$\beta_2^n = (C_3C_4 - C_1C_5)/(C_3C_3 - C_1C_2) \tag{10b}$$

where

$$C_1 = \int_{t=0}^{t_f} [\Delta\bar{T}(x_1,t)^2 + \Delta\bar{T}(x_2,t)^2]\, \mathrm{d}t \tag{10c}$$

$$C_2 = \int_{t=0}^{t_f} [\Delta\tilde{T}(x_1,t)^2 + \Delta\tilde{T}(x_2,t)^2]\, \mathrm{d}t \tag{10d}$$

$$C_3 = \int_{t=0}^{t_f} [\Delta\bar{T}(x_1,t) \times \Delta\tilde{T}(x_1,t) + \Delta\bar{T}(x_2,t) \times \Delta\tilde{T}(x_2,t)]\, \mathrm{d}t \tag{10e}$$

$$C_4 = \int_{t=0}^{t_f} \{[T(x_1,t) - Y(x_1,t)]\Delta\bar{T}(x_1,t) + [T(x_2,t) - Y(x_2,t)]\Delta\bar{T}(x_2,t)\}\, \mathrm{d}t \tag{10f}$$

$$C_5 = \int_{t=0}^{t_f} \{[T(x_1,t) - Y(x_1,t)]\Delta\tilde{T}(x_1,t) + [T(x_2,t) - Y(x_2,t)]\Delta\tilde{T}(x_2,t)\}\, \mathrm{d}t \tag{10g}$$

B. Adjoint Problem and Gradient Equation

To obtain the adjoint problem, Eq. (1a) is multiplied by a Lagrange multiplier (or adjoint function) $\lambda_1(x,t)$. The resulting expression is integrated over the correspondent space and time domains, and then the result is added to the right-hand side of Eq. (2) to yield the following expression for the functional $J[q_1(t), q_2(t)]$:

$$J[q_1(t), q_2(t)] = \int_{t=0}^{t_f} \{[T(x_1,t) - Y(x_1,t)]^2 + [T(x_2,t) - Y(x_2,t)]^2\}\, \mathrm{d}t + \int_{t=0}^{t_f}\int_{x=0}^{L} \lambda_1(x,t) \times \left[k\frac{\partial^2 T}{\partial x^2} - \tau\rho c\frac{\partial^2 T}{\partial t^2} - \rho c\frac{\partial T}{\partial t}\right] \mathrm{d}x\, \mathrm{d}t \tag{11}$$

The variation ΔJ_1 is obtained by replacing $q_1(t)$ by $q_1(t) + \Delta q_1(t)$ and T by $T + \Delta\bar{T}(x,t)$ in Eq. (11), subtracting the original Eq. (11) from the resulting expression, and neglecting the second-order terms. It thus finds

$$\Delta J_1[q_1(t), q_2(t)] = \int_{t=0}^{t_f} \{2[T(x_1,t) - Y(x_1,t)]\Delta\bar{T} + 2[T(x_2,t) - Y(x_2,t)]\Delta\bar{T}\}\, \mathrm{d}t + \int_{t=0}^{t_f}\int_{x=0}^{L} \lambda_1(x,t) \times \left[k\frac{\partial^2 \Delta\bar{T}}{\partial x^2} - \tau\rho c\frac{\partial^2 \Delta\bar{T}}{\partial t^2} - \rho c\frac{\partial \Delta\bar{T}}{\partial t}\right] \mathrm{d}x\, \mathrm{d}t \tag{12a}$$

which can be rearranged as

$$\Delta J_1[q_1(t), q_2(t)] = \int_{t=0}^{t_f}\int_{x=0}^{L} \{2[T(x,t) - Y(x,t)]\Delta\bar{T}[\delta(x - x_1) + \delta(x - x_2)]\}\, \mathrm{d}x\, \mathrm{d}t + \int_{t=0}^{t_f}\int_{x=0}^{L} \lambda_1(x,t) \times \left[k\frac{\partial^2 \Delta\bar{T}}{\partial x^2} - \tau\rho c\frac{\partial^2 \Delta\bar{T}}{\partial t^2} - \rho c\frac{\partial \Delta\bar{T}}{\partial t}\right] \mathrm{d}x\, \mathrm{d}t \tag{12b}$$

where $\delta(\cdot)$ is the Dirac delta function.

In Eq. (12b), the integrands containing adjoint function $\lambda_1(x,t)$ on the right-hand side are integrated by parts; the initial and boundary conditions of the sensitivity problem (6) are used. The vanishing of the integrands leads to the following adjoint problem for the determination of $\lambda_1(x,t)$:

$$k\frac{\partial^2\lambda_1}{\partial x^2} = \tau\rho c\frac{\partial^2\lambda_1}{\partial t^2} - \rho c\frac{\partial\lambda_1}{\partial t} - 2[T(x,t) - Y(x,t)][\delta(x-x_1) + \delta(x-x_2)]; \qquad 0<x<L, \qquad t>0 \tag{13a}$$

$$\lambda_1(x,t_f) = 0; \qquad 0<x<L, \qquad t=t_f \tag{13b}$$

$$\frac{\partial\lambda_1(x,t_f)}{\partial t} = 0; \qquad 0<x<L, \qquad t=t_f \tag{13c}$$

$$\frac{\partial\lambda_1(0,t)}{\partial x} = 0; \qquad x=0, \qquad t>0 \tag{13d}$$

$$\frac{\partial\lambda_1(L,t)}{\partial x} = 0; \qquad x=L, \qquad t>0 \tag{13e}$$

The adjoint problem differs from the standard initial value problems in that the final time conditions at time $t=t_f$ is specified instead of the customary initial condition. However, this problem can be transformed to an initial value problem by the transformation of the time variables as $\eta = t_f - t$. Then the Crank–Nicolson type central difference discretization [4] can be used to solve the preceding adjoint problem.

Finally, the following integral term is left

$$\Delta J_1 = \int_{t=0}^{t_f}\left[\lambda(0,t) - \tau\lambda(0,t)\delta(t-0) - \tau\frac{\partial\lambda(0,t)}{\partial t}\right]\Delta q_1(t)\,\mathrm{d}t \tag{14a}$$

From definition [13], the functional increment can be presented as

$$\Delta J_1 = \int_{t=0}^{t_f} J_1'[q_1(t)]\Delta q_1(t)\,\mathrm{d}t \tag{14b}$$

A comparison of Eqs. (14a) and (14b) leads to the following expression for determining the gradient of functional $J_1'[q_1(t)]$:

$$J_1'[q_1(t)] = \lambda_1(0,t) - \tau\lambda_1(0,t)\delta(t-0) - \tau\frac{\partial\lambda_1(0,t)}{\partial t} \tag{15}$$

Similarly, to derive the adjoint problem for the case when perturbing $q_2(t)$, Eq. (1a) is multiplied by a second Lagrange multiplier (or adjoint function) $\lambda_2(x,t)$. By following the same procedure as described previously, it is found that the solutions for the adjoint equation $\lambda_2(x,t)$ are identical to that for $\lambda_1(x,t)$. This implies that the adjoint equations need to be solved only once because $\lambda_1(x,t) = \lambda_2(x,t)$. Finally, the gradient equation for $q_2(t)$ can be obtained as

$$J_2'[q_2(t)] = \lambda_1(L,t) - \tau\lambda_1(L,t)\delta(t-0) - \tau\frac{\partial\lambda_1(L,t)}{\partial t} \tag{16}$$

C. Stopping Criterion

If the problem contains no measurement errors, the traditional check condition is specified as

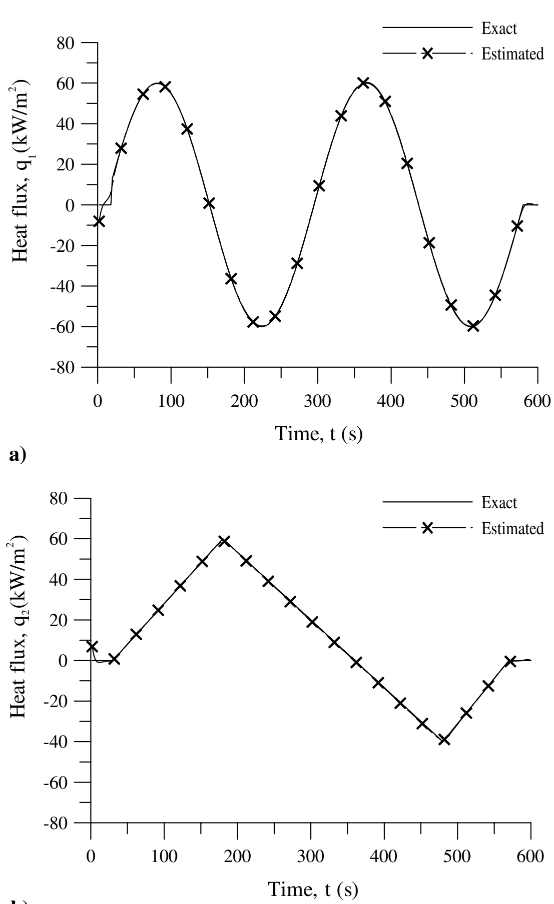

Fig. 1 The exact and estimated a) $q_1(t)$ and b) $q_2(t)$ with $x_1 = 0.0035$ m and $x_2 = 0.0315$ m using $\sigma = 0$ in test case 1.

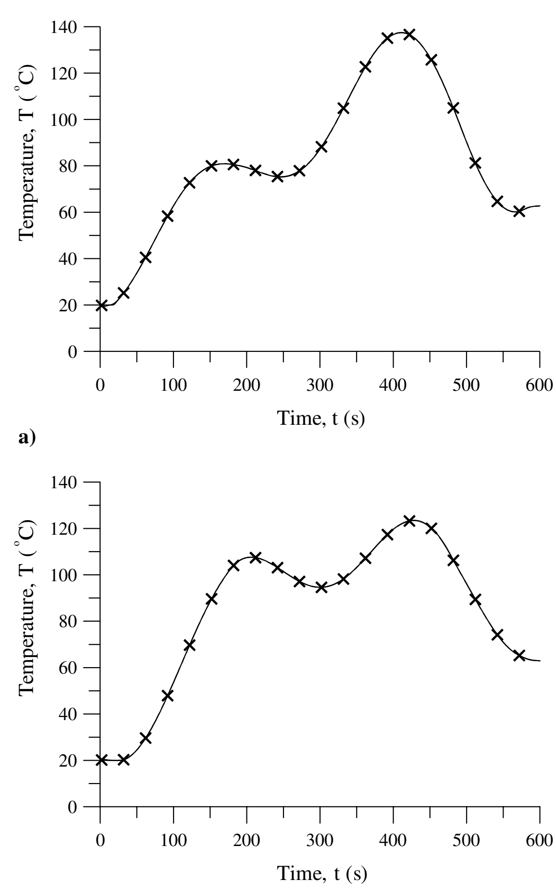

Fig. 2 The measured and estimated temperatures at a) $x_1 = 0.0035$ m and b) $x_2 = 0.0315$ m with time using $\sigma = 0$ in test case 1.

$$J[q_1^{n+1}(t), q_2^{n+1}(t)] < \varepsilon \tag{17a}$$

where ε is a small-specified number. However, the measured temperature data must contain measurement errors. Therefore, it is not expected that the functional Eq. (2) be equal to zero at the final iteration step. Following the experiences of the authors [13–19], the discrepancy principle can be used as the stopping criterion, that is, it is assumed that the temperature residuals may be approximated by

$$T(x_1, t) - Y(x_1, t) = T(x_2, t) - Y(x_2, t) \approx \sigma \tag{17b}$$

where σ is the standard deviation of the measurements, which is assumed to be a constant. Substituting Eq. (17b) into Eq. (2), the following expression is obtained for stopping criteria ε:

$$\varepsilon = 2\sigma^2 t_f \tag{17c}$$

Then, the stopping criterion is given by Eq. (17a) with ε determined from Eq. (17c).

V. Computational Procedure

The computational procedure for the solution of this inverse hyperbolic heat conduction problem using the CGM may be summarized as follows:

Suppose $q_1^n(t)$ and $q_2^n(t)$ are available at iteration n.

Step 1) Solve the direct problem given by Eq. (1) for $T(x, t)$.

Step 2) Examine the stopping criterion given by Eq. (17a) with ε given by Eq. (17c). Continue if not satisfied.

Step 3) Solve the adjoint problem given by Eqs. (13) for $\lambda_1(x, t)$.

Step 4) Compute the gradient equations of the functional $J_1'^n(t)$ and $J_2'^n(t)$ from Eqs. (15) and (16), respectively.

Step 5) Compute the conjugate coefficients γ_1^n and γ_2^n and directions of descent $P_1^n(t)$ and $P_2^n(t)$ from Eqs. (5) and (4), respectively.

Step 6) Set $\Delta q_1(t) = P_1^n(t)$ and $\Delta q_2(t) = P_2^n(t)$, and then solve the sensitivity problems given by Eqs. (6) and (7) for $\Delta\bar{T}(x, t)$ and $\Delta\tilde{T}(x, t)$, respectively.

Step 7) Compute the search step sizes β_1^n and β_2^n from Eq. (10).

Step 8) Compute the new estimations for $q_1^{n+1}(t)$ and $q_2^{n+1}(t)$ from Eqs. (3a) and (3b) and return to step 1.

VI. Results and Discussion

The goal of this work is to verify the inverse algorithm by the CGM in estimating simultaneously two unknown boundary heat fluxes $q_1(t)$ and $q_2(t)$ accurately in a hyperbolic conduction model with no prior information on the functional form of the unknown quantities.

The accuracy test of the numerical solution for the direct problem has been done in [19], in which a benching mark problem [4] has been considered and solved. The conclusions verified the correctness of the numerical solution for this hyperbolic conduction problem.

To illustrate the accuracy of the CGM in predicting boundary heat fluxes $q_1(t)$ and $q_2(t)$ with the present inverse analysis from the knowledge of transient temperature recordings at two sensors, two specific examples with different forms of heat fluxes, different measurement positions, and different measurement errors are considered.

To compare the results for situations involving random measurement errors, a normally distributed uncorrelated error with

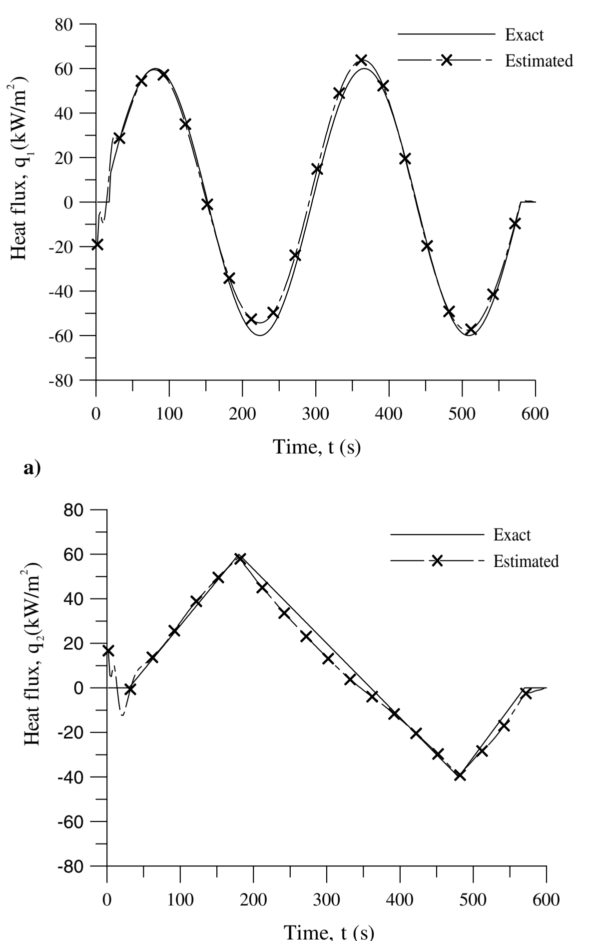

Fig. 3 The exact and estimated a) $q_1(t)$ and b) $q_2(t)$ with $x_1 = 0.0105$ m and $x_2 = 0.0245$ m using $\sigma = 0$ in test case 1.

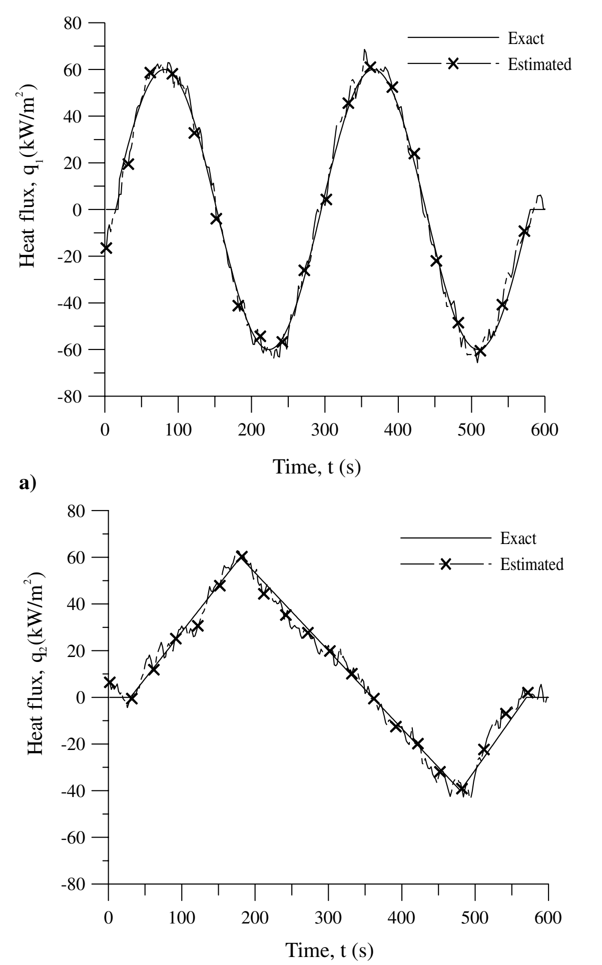

Fig. 4 The exact and estimated a) $q_1(t)$ and b) $q_2(t)$ with $x_1 = 0.0035$ m and $x_2 = 0.0315$ m using $\sigma = 2.5$ in test case 1.

zero mean and constant standard deviation are considered. The simulated inexact measurement data Y can be expressed as

$$Y = Y_{\mathrm{dir}} + \omega\sigma \tag{18}$$

where Y_{dir} is the solution for the direct problem with exact boundary heat fluxes $q_1(t)$ and $q_2(t)$, σ is the standard deviation of the measurement error, and ω is a random variable that is generated by the subroutine DRNNOR of the International Mathematical and Statistical Library (IMSL) [21] and will be within -2.576 to 2.576 for a 99% confidence bound.

One of the advantages of using the CGM to solve the inverse problems is that the initial guesses of the unknown quantities can be chosen arbitrarily. In all the test cases considered here, the initial guesses of $q_1(t)$ and $q_2(t)$ are taken as $q_1^0(t) = q_2^0(t) = 0.0$.

The following computational parameters are chosen for the numerical experiments [19]:

$$T_0 = 20°\mathrm{C}; \qquad L = 0.035\ \mathrm{m}; \qquad k = 50\ \mathrm{W \cdot m^{-2}};$$
$$\alpha = k/\rho c = 1.327 \times 10^{-5}\ \mathrm{m^2 s^{-1}}$$

Here, α is the thermal diffusivity of the material. The space and time increments used in numerical calculations are taken as $\Delta x = 0.00035$ m (i.e., 100 grid points in space) and $\Delta t = 2.0$ s (i.e., 300 grid points for $t_f = 600$ s). We now present the following two numerical test cases in determining simultaneously $q_1(t)$ and $q_2(t)$ by the inverse analysis using the CGM.

A. Numerical Test Case 1

The unknown transient boundary heat fluxes $q_1(t)$ and $q_2(t)$ are assumed applied at $x = 0$ and L in the following forms:

$$q_1(t) = \begin{cases} 0; & 0 \le t \le 18\ \mathrm{s} \\ 6 \times 10^4 \times \sin\left[\frac{4\pi(t-10)}{570}\right] \frac{\mathrm{W}}{\mathrm{m}^2}; & 18 < t \le 580\ \mathrm{s} \\ 0; & 580 < t \le 600\ \mathrm{s} \end{cases} \tag{19a}$$

$$q_2(t) = \begin{cases} 0; & 0 \le t < 30\ \mathrm{s} \\ \frac{3\times10^4}{75} t - \frac{90\times10^4}{75}; & 30 \le t < 180\ \mathrm{s} \\ -\frac{5\times10^4}{150} t + \frac{1800\times10^4}{150} \frac{\mathrm{W}}{\mathrm{m}^2}; & 180 \le t < 480\ \mathrm{s} \\ \frac{2\times10^4}{45} t - \frac{1140\times10^4}{45}; & 480 \le t < 570\ \mathrm{s} \\ 0; & 570 \le t < 600\ \mathrm{s} \end{cases} \tag{19b}$$

The inverse analysis is first performed by using $\tau = 1.0$ s, $\Delta t = 2.0$ s, $x_1 = 0.0035$ m, $x_2 = 0.0315$ m and assuming exact measurements $\sigma = 0.0$. By choosing stopping criterion $\varepsilon = 300$, the exact and estimated $q_1(t)$ and $q_2(t)$ after 16 iterations are shown in Figs. 1a and 1b, respectively. It can be seen from Fig. 1 that the estimated heat fluxes are both very accurate and no phase errors have been observed. In addition, in Yang's study [12], some future time measurements are required, but they are not needed in the present algorithm and, therefore, the advantage of using the CGM is thus proven. The measured and estimated temperatures at $x_1 = 0.0035$ m and $x_2 = 0.0315$ m are plotted in Figs. 2a and 2b, respectively.

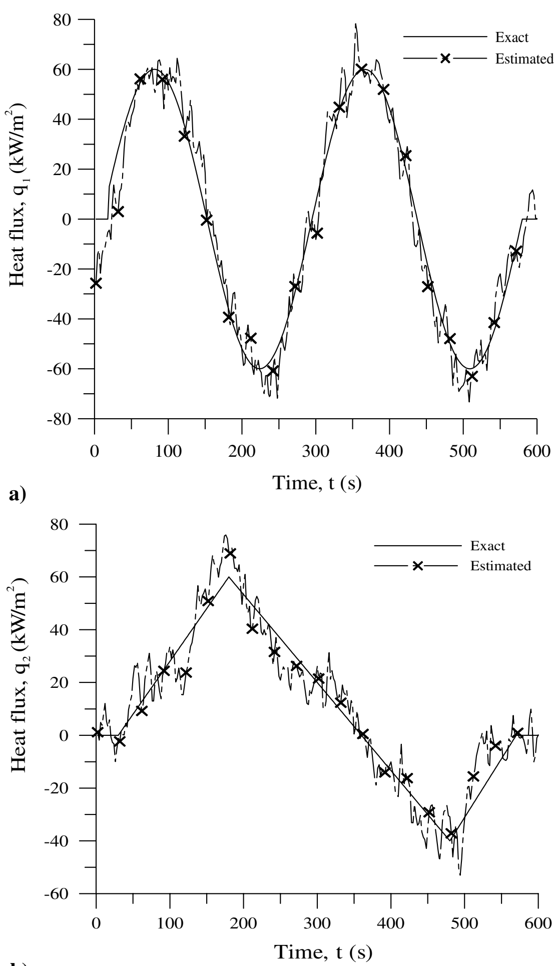

Fig. 5 The exact and estimated a) $q_1(t)$ and b) $q_2(t)$ with $x_1 = 0.0035$ m and $x_2 = 0.0315$ m using $\sigma = 6.7$ in test case 1.

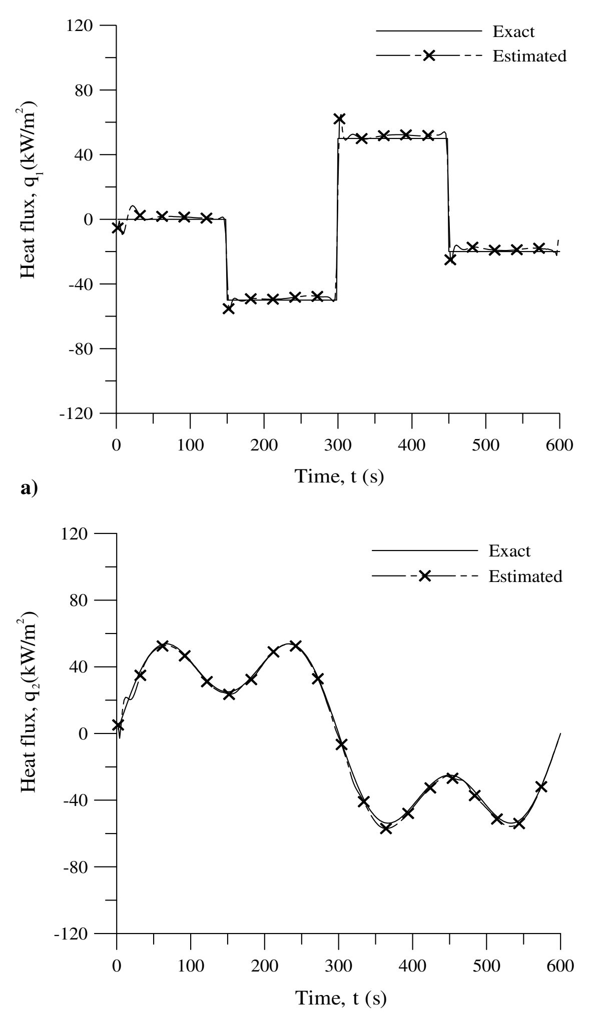

Fig. 6 The exact and estimated a) $q_1(t)$ and b) $q_2(t)$ with $x_1 = 0.0035$ m and $x_2 = 0.0315$ m using $\sigma = 0$ in test case 2.

The average relative errors for estimated heat fluxes and temperatures are calculated as ERR1 = 2.255%, ERR2 = 2.653%, ERR3 = 0.436%, and ERR4 = 0.571%, respectively, where the relative average errors for the estimated $q_1(t)$, $q_2(t)$, $T(x_1, t)$, and $T(x_2, t)$ are defined as

$$\mathrm{ERR}\,1\% = \left[\sum_{J=1}^{M}\left|\frac{q_1(J) - \hat{q}_1(J)}{q_1(J)}\right|\right] \div M \times 100\% \tag{20a}$$

$$\mathrm{ERR}\,2\% = \left[\sum_{J=1}^{M}\left|\frac{q_2(J) - \hat{q}_2(J)}{q_2(J)}\right|\right] \div M \times 100\% \tag{20b}$$

$$\mathrm{ERR}\,3\% = \left[\sum_{J=1}^{M}\left|\frac{T(x_1, J) - Y(x_1, J)}{Y(x_1, J)}\right|\right] \div M \times 100\% \tag{20c}$$

$$\mathrm{ERR}\,4\% = \left[\sum_{J=1}^{M}\left|\frac{T(x_2, J) - Y(x_2, J)}{Y(x_2, J)}\right|\right] \div M \times 100\% \tag{20d}$$

Here, J represents the index of discreted time, whereas $\hat{q}_1(J)$, $\hat{q}_2(J)$ and $T(x_1, J)$, $T(x_2, J)$ denote the estimated values of heat fluxes and temperatures, respectively. Some adjacent points to the zero value for both heat fluxes and temperatures are neglected in calculating the relative error to avoid unreasonable computations.

Next, it is of interest to discuss what will happen when the measured positions are changed, that is, measurement positions are now at $x_1 = 0.0105$ m and $x_2 = 0.0245$ m, that is, both sensors are moved toward to the center of the domain. The same calculation conditions are used and the number of iterations under this situation is 165 with $\varepsilon = 300$.

The estimated $q_1(t)$ and $q_2(t)$ are plotted in Figs. 3a and 3b, respectively, and ERR1 to ERR4 are obtained as ERR1 = 4.998%, ERR2 = 3.519%, ERR3 = 0.581%, and ERR4 = 0.652%. It is obvious that the estimated $q_1(t)$ and $q_2(t)$ become worse and the number of iterations is also increased. This implies that two sensors should be placed as close to each boundary as possible for the present inverse problem. The reason is that when the sensors are moved to the center, the differences of the temperature readings in each sensor will become smaller, and this makes the algorithm difficult to estimate both unknown heat fluxes simultaneously. When both sensors are placed at the center of the domain, the readings are identical to each other and it is impossible to estimate two heat fluxes at the same time.

Finally, let us discuss the influence of the measurement errors on the inverse solutions. First, the measurement error for the temperatures measured by sensors is taken as $\sigma = 2.5$ (about 3% of the average measured temperatures at x_1 and x_2) with $x_1 = 0.0035$ m and $x_2 = 0.0315$ m. After 13 iterations, the estimated $q_1(t)$ and $q_2(t)$ can be obtained and are plotted in Figs. 4a and 4b, respectively. The relative errors ERR1 and ERR2 are calculated as ERR1 = 6.360% and ERR2 = 7.385%, respectively. Second, the measurement error is increased to $\sigma = 6.7$ (about 8% of the average measured temperatures at x_1 and x_2). After only 12 iterations, the estimated $q_1(t)$ and $q_2(t)$ can be obtained and plotted in Figs. 5a and

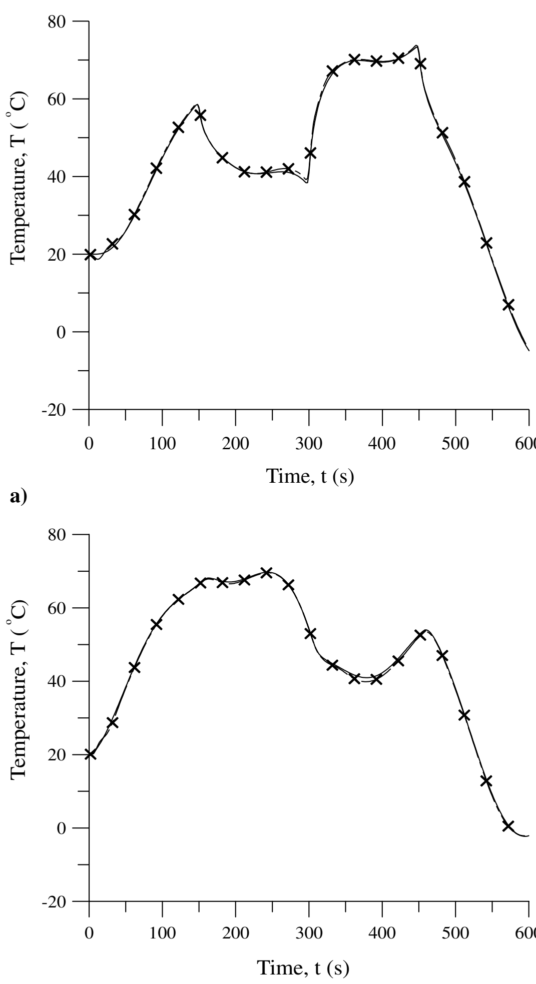

Fig. 7 The measured and estimated temperatures at a) $x_1 = 0.0035$ m and b) $x_2 = 0.0315$ m with time using $\sigma = 0$ in test case 2.

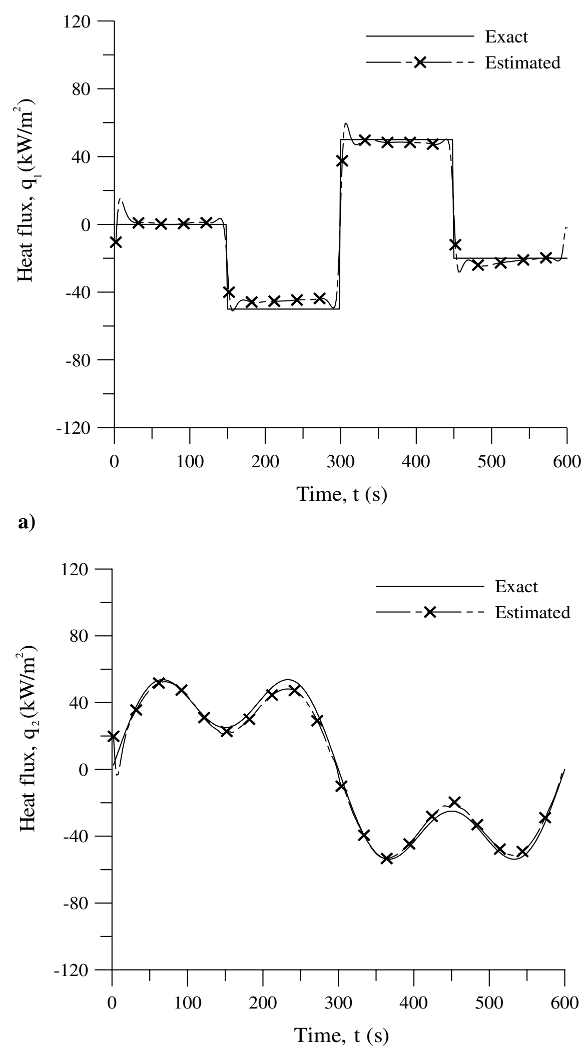

Fig. 8 The exact and estimated a) $q_1(t)$ and b) $q_2(t)$ with $x_1 = 0.0105$ m and $x_2 = 0.0245$ m using $\sigma = 0$ in test case 2.

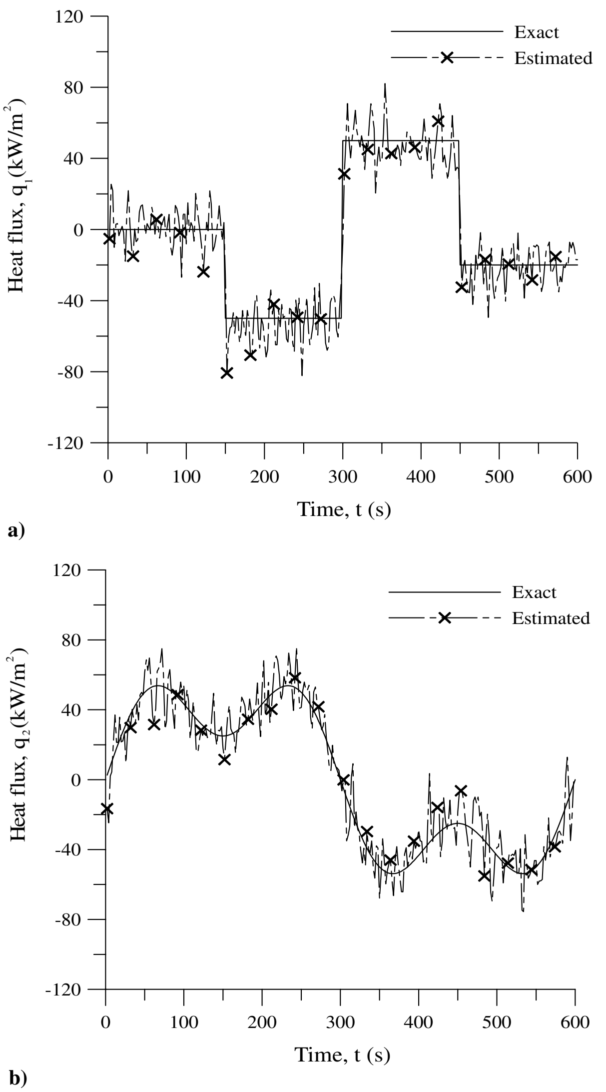

Fig. 9 The exact and estimated a) $q_1(t)$ and b) $q_2(t)$ with $x_1 = 0.0035$ m and $x_2 = 0.0315$ m using $\sigma = 1.4$ in test case 2.

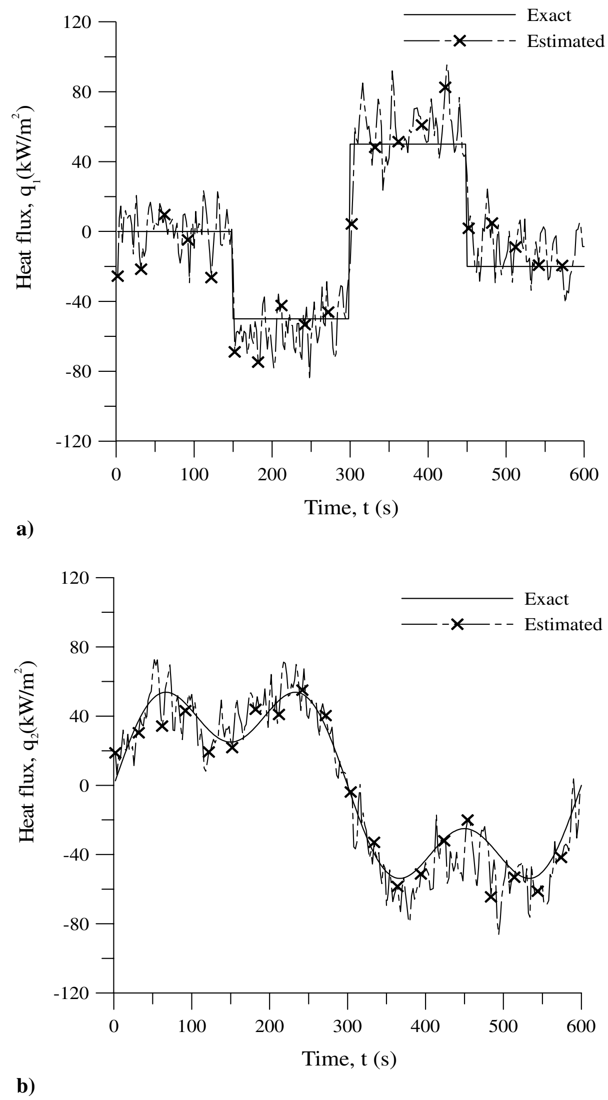

Fig. 10 The exact and estimated a) $q_1(t)$ and b) $q_2(t)$ with $x_1 = 0.0035$ m and $x_2 = 0.0315$ m using $\sigma = 3.6$ in test case 2.

5b, respectively. The relative errors ERR1 and ERR2 are calculated as ERR1 = 9.320% and ERR1 = 10.826%, respectively. From those results we learned that the reliable inverse solutions can still be obtained when large measurement errors are considered.

B. Numerical Test Case 2

The unknown boundary heat fluxes $q_1(t)$ and $q_2(t)$ are assumed as the following functions:

$$q_1(t) = \begin{cases} 0; & 0 \le t < 150 \text{ s} \\ -5 \times 10^4; & 150 \le t < 300 \text{ s} \\ 5 \times 10^4 \frac{\text{W}}{\text{m}^2}; & 300 \le t < 450 \text{ s} \\ -2 \times 10^4; & 450 \le t < 600 \text{ s} \end{cases} \tag{21a}$$

$$q_2(t) = 5 \times 10^4 \left[\sin\left(\frac{\pi t}{600}\right) + 0.5 \sin\left(\frac{3\pi t}{600}\right) \right] \frac{\text{W}}{\text{m}^2}; \quad 0 \le t < 600 \text{ s} \tag{21b}$$

By using $\tau = 1.0$ s, $\Delta t = 2.0$ s, $x_1 = 0.0035$ m, $x_2 = 0.0315$ m and $\sigma = 0.0$. When considering stopping criterion $\varepsilon = 300$, the exact and estimated $q_1(t)$ and $q_2(t)$ after 88 iterations are plotted in Figs. 6a and 6b, respectively, whereas the measured and estimated temperatures at $x_1 = 0.0035$ m and $x_2 = 0.0315$ m are plotted in Figs. 7a and 7b, respectively. The average relative errors for estimated heat fluxes and temperatures are calculated as ERR1 = 3.364%, ERR2 = 3.658%, ERR3 = 1.437%, and ERR4 = 1.488%, respectively.

Next, the measured positions are changed to $x_1 = 0.0105$ m and $x_2 = 0.0245$ m. With the same calculation conditions and after 77 iterations, the estimated $q_1(t)$ and $q_2(t)$ are shown in Figs. 8a and 8b, respectively. The relative errors for ERR1 to ERR4 are calculated as ERR1 = 4.628%, ERR2 = 6.387%, ERR3 = 1.401%, and ERR4 = 0.737%. The estimated $q_1(t)$ and $q_2(t)$ still become worse, and this implies that the sensors should be placed closer to the boundary.

Finally, the influences of the measurement errors on the inverse solutions are examined. The measurement error is first taken as $\sigma = 1.4$ (about 3% of the average measured temperatures at x_1 and x_2) with $x_1 = 0.0035$ m and $x_2 = 0.0315$ m. After 71 iterations, the estimated $q_1(t)$ and $q_2(t)$ are obtained and plotted in Figs. 9a and 9b, respectively. The relative errors for ERR1 and ERR2 are calculated as ERR1 = 6.748% and ERR1 = 8.888%, respectively. Then, the measurement error is increased to $\sigma = 3.6$ (about 8% of the average measured temperatures at x_1 and x_2). After only 25 iterations, the estimated $q_1(t)$ and $q_2(t)$ are obtained and plotted in Figs. 10a and 10b, respectively. The relative errors for ERR1 and ERR2 are calculated as ERR1 = 8.296% and ERR1 = 12.682%, respectively.

From the preceding two test cases, it can be seen that the present inverse problems in estimating simultaneously two time-dependent boundary heat fluxes are now completed. Reliable estimations can be obtained without any phase errors when using either exact or error measurements.

VII. Conclusions

The conjugate gradient method was successfully applied for the solution of the inverse hyperbolic heat conduction problem in determining simultaneously two unknown boundary heat fluxes by using the simulated temperature readings taken inside the medium. Two test cases involving different heat fluxes, measured positions, and measurement errors were examined. The results show that the measured positions should be close to the boundary for accurate estimations, and inverse solutions obtained by the CGM remain stable and regular as the measurement errors are increased.

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